The Incan Abacus: A Curious Counting Device

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Abstract:

Little is known about the ancient calculating device of the Incans la yupana. It was a tablet upon which stones, grains, or beans were placed and manipulated to perform calculations. This paper critically examines the little information available on la yupana and proposes a new interpretation, taking into consideration both mathematical and cultural factors. The conclusion reached is that la yupana is an abacus that directly correlates with the quipus, performs addition, subtraction, multiplication, and division in base ten, can be used to decompose numbers, and reflects linguistic principles of the Aymara language. A better understanding of the Incan abacus could lead to further insights about the civilization.

Introduction

La Yupana

La yupana[1] was the ancient calculating device of the Incas. In the indigenous Andean language Quechua "*yupar*" means "to count." There are numerous historical accounts that mention the existence of such a calculating device, but the way they worked is left very unclear. What we do know for sure is that the Incas used some sort of tablet upon which stones or beans were placed. These markers were moved

Figure 1, from Day, page 35.

[1] *La yupana* and "abacus" will be used interchangeably throughout the essay.

around in order to perform the various calculations needed to make the *quipu* records.[2] Both the *quipus* and *la yupana* are shown in the image to the right. The *quipucamoyoc* is holding the *quipus* while the abacus is located in the lower left corner (see Figure 1). It would not have been very feasible to tie and untie the knots on the *quipus* itself, hence, the versatility the abacus provides would have been essential. Stones or other markers were used because they could easily be moved and relocated. Once the operation was performed on the abacus the final computation could then be tied permanently into the *quipus*. It is easily seen that some sort of calculating device was used, but much more difficult to identify exactly how *la yupana* was used.

The most important piece of information in solving this riddle is the sketch referred to previously by Guaman Poma de Ayala, a chronicler of the 16th century**.** While a picture is worth a thousand words, I really wish that Poma de Ayala had left 1000 words describing exactly how *la yupana* worked. All that is available is this single picture and a few brief written accounts. That's not to say that we are unable to know anything of interest about this counting mechanism- it will just require more work and diligence on our part. Much can be inferred about this counting device while examining it within a comprehensive overview of the Incan way of life. This essay will introduce a few possible interpretations of this Incan calculating device, and determine the best one (or combination thereof). Because we are analyzing a math tool from a very distinct time period and culture, all the theories should be criticized accordingly. The two most important factors to consider are: (a) the ease of manipulating the numbers and making

[2] The *quipus* is an artifact of the Incan civilization that is directly related to *la yupana*. It is a knot-tying record composed of pendant cords hanging from a main cord. The number and types of knots on each pendant cord determine the numbers that are represented. These numbers denote all the tracked quantities of the empire from potato inventory to taxes paid to the Sapa Inca.

calculations and (b) the consistency with Incan culture and their perception of numbers. The most cogent solution is the one that makes the most sense mathematically as well as culturally.

Historical Accounts

The first attempted solution was the work published in 1931 by Henry Wassén. His article "The Ancient Peruvian Abacus" appeared in Nordenskiold (1931), a collection of comparative ethnographical studies. His solution is based on two written accounts of Incan arithmetic. The first of these accounts is the manuscript accompanying Poma de Ayala's sketch of the abacus. Ayala writes:

> "Haciendan Chasquicoc, the main treasurer, says that [Tahuantisuyo Runa Quipoc Incap, the main accountant] had great ability. In order to determine his skill, the Inca sent him to count and adjust Indian accounts in this region. . . His skill was greater than it would have been with pen and paper. [The counters] count with tablets. They number 100,000 and 10,000 and 100 and 10 until arriving at 1. They count everything that goes on in the kingdom: the parties, the Sundays, the months, the years, and these eight treasurers and accountants were in every city, village, and Indian town of the kingdom. They counted in this way: starting with one, two and three- 1 – 2 – 3 – 4 – 5 – 6 – 7 – 8 – 9 – 10 – 20 – 30 – 40 – 50 – 60 – 70 – 80 – 90 – 100 – 1000 – 1000 – 100000 – 1000000 – 10000000 – infinity."[3]

The second reference used is that of Jose de Acosta taken from "*La Historia Natural y Moral de Las Indias*" originally published in 1586:

> "To see them use another kind of *quipu* with maize kernels is a perfect joy. In order to effect a very difficult computation for which an able calculator would require pen and ink for the various methods of calculation these Indians make use of their kernels. They place one here, three somewhere else and eight I do not know where. They move one kernel here and three there and the fact is that they are able to complete their computation without making the smallest mistake. As a matter of fact, they are better at calculating what each one is due to pay or give than we should be with pen and ink. Whether this is not ingenious and whether these people are wild animals let those judge who will! What I consider as certain is that in what they undertake to do they are superior to us."[4]

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[3] Ayala quoted in Burns, page 60, my translation
[4] Acosta, translation

From these sources, Wassén deduces that (1) the operator was very skilled (2) he was commanded by the Sapa Inca to perform these calculations throughout the kingdom (3) he used wool and *quinoa*[5] and could compute more cleverly than those with pen and paper. Many words from the indigenous Andean language of Aymara allude to these types of computation. Excerpts taken from Bertonio's Spanish-aymara dictionary "Vocabulario de la Lengua Aymara" are evidence of this. The Quechua words *calana, aponocatha, iranocatha, saraatha*, and *velinocatha* all mean "to count with little rocks."[6] *Cchaara* is a "rock count to show what is due" and *hanka* is "what has been paid."[7] Other evidence suggests that these operations were endemic to Ecuador and performed by other indigenous groups as well:

> "The archives reduced down to wooden, rock, or clay deposits with many compartments in which rocks of all sizes, colors, and angles were placed; they were excellent lapidaries. With the diverse combinations of the rocks, they perpetuated the events and created accounts of everything."[8]

Wassén's Solution

The synthesis of this information led Wassén to the following solution. In this interpretation *la yupana* serves as a coordinate system with the vertical columns based on the decimal system and the numerical values of the horizontal rows "based on the important part that the number 5, i.e. a hand, plays among many people, and evidently played in the development of the decimal system in Peru."[9] Addition would be a simple enough computation. If you start with three stones in Ae for example and would like to add three more, you would add two more stones to Ae (until that square is full), replace those stones by one stone in Be and the last stone

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[5] A very important grain and staple for the Incans
[6] Bertonio, Page 139, my translation
[7] Bertonio, Page 367, my translation
[8] Wassen in Nordenskiold
[9] Wassen in Nordenskiold, page 200

would be placed in Ae. Wassén argues that any summation can henceforth be directly related to the *quipus*. The problem with this idea is that if this were the case, the only holes needed in the abacus would be column A and one of the holes in each of the boxes of column B. Such an abacus could sufficiently hold all values. Wassén concedes that this is the case, but that columns C and D would be necessary for values larger than 222,220. It is a very improbable conjecture. (Especially since the Incas would rarely need to make such large calculations- the Ayala manuscript even implies that on the computing boards they numerated from one to 100,000; even if the Incas did work with larger numbers, it is even more unlikely that they would change the whole structure and functionality of their computing device in order to accommodate for the larger numbers.)

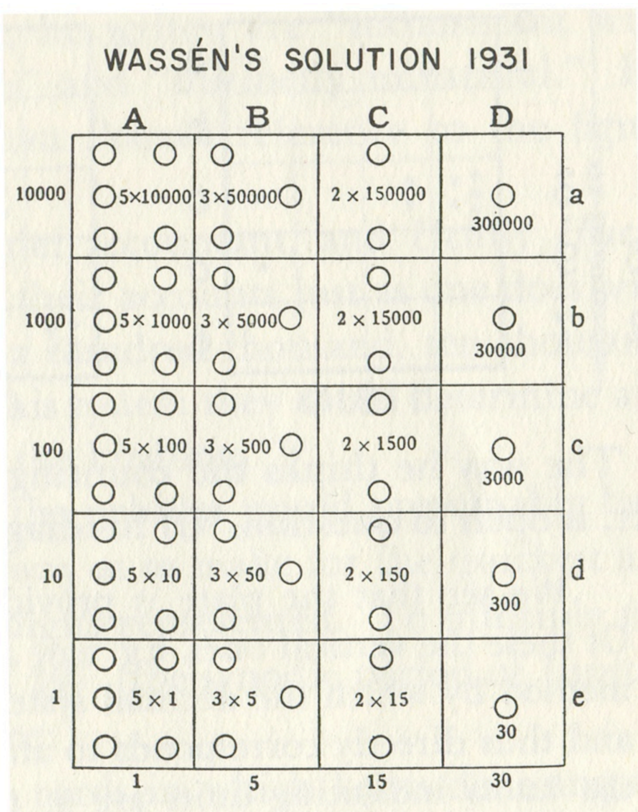


Figure 2, from *Day*, page 35.

Wassén's theory is indeed representative of a possible way to value and arrange the rocks on *la yupana*, but is far from likely. This type of number manipulation is neither derived easily nor is very natural or intuitive. Even with a strong background in mathematics, one would find it difficult to conceive and effectively utilize such a method, with so many unnecessary holes and place values. There is a definite gap in logic and the cultural significance of such a board structure is just as underdeveloped. Wassén is not able to support his claims. "We find that the solitary hollows in row D, counting from the bottom, denote the values of 30, 300, 3000 and so on. The importance of these squares in chronological computation by for instance setting down ten years as equaling 3650 days is no doubt self-evident."[10] I would argue that making such an unfounded jump is the only thing that is self-evident here.

Symbolic Values

Another possibility that has surfaced but has come up short from fully explaining the enigma, is that proposed by JA Mason and Diego Pareja. Pareja (1986) notes that many pre-Hispanic civilizations have made use of symbolic notation to represent different numerical values. The Babylonians used vertical and horizontal hooks and wedges, the Chinese used vertical and horizontal lines, and the Mayans were able to express all mathematical values with only three symbols: a dot, a bar and a shell-like symbol (representing zero). Applying the same logic, Pareja proposes that the Incas employed two symbols to denote values: a white circle representing "one" and a black circle representing "five." The distinction would be made by using stones of two different colors. Every number could be represented by at most five stones and an empty spot in the abacus would be understood as a zero. This explanation is clearly lacking because such a system has no apparent correlation between stone values and the differing number of holes in each square of the abacus. Considering that each square has a different number of holes (one, two, three, or five), blank squares represent zero, and each stone has a different face value (either one or five), the values of each *yupana* square could range from zero to 25. This lack of uniformity doesn't seem very plausible.

Pasquale's Solution

[10] Wassen in Nordenskiold , page 201

More recently in 2001, an Italian engineer came up with a solution that is uniform and does indeed take into consideration and utilizes the fact that the tablet squares have various numbers of holes; the problem is that this solution is not in conformity with the cultural norms and characteristics of the Incan culture and civilization. Nicolino De Pasquale proposes that *la yupana* actually calculated in base 40. The stone in the square with one hole would have value one, the two stones in the square with two holes would each have value two, the three stones in the square with three holes would each have value three, and finally the five stones in the square with five holes would each have value five. This yields a total of 1+2(2)+3(3)+5(5) = 39. The values one through 39 could all be represented by placing the corresponding number of stones in the first row; once this row was filled, one stone would be placed in the second row, indicating 40, and the process would continue. While this method is indeed more structured and logical than the previous proposals, it likewise lacks depth of content. De Pasquale claims "It took me about 40 minutes to solve the riddle. I am not an expert on pre-Columbian civilizations. I simply decoded a 16th century drawing from a book on mathematical enigmas I received as a Christmas present." Just because something seems plausible from a mathematical perspective, does not mean that it is meaningful with regards to the cultural context. No matter how brilliant a solution might seem, it should not be accepted definitively before a rigorous analysis and understanding of the context from which the problem stems; this cannot possibly be done in 40 minutes.

Alternate Interpretation

It is very easy to criticize hypotheses and accentuate shortcomings, but it is a much more difficult feat to propose alternatives. Considering *la yupana* in light of *The Social Life of Numbers: A Quechua Ontology of Numbers and Philosophy of Arithmetic* by Gary Urton (1997), some interesting possibilities emerge. Originally, I considered turning the abacus on its side so that the row of ones would be the first row. This perspective lends itself to a very unique interpretation. The five ones could be correlated to the hand, each representing a finger, with value one. Once these five holes were filled, you would have successfully completed a "hand." This interpretation has potential because it applies the Incan notion of completion. Incas were very aware of the "complete" state of numbers and strived to make numbers more complete. Complete numbers (such as paired or even numbers) were always valued over incomplete numbers (such as odd numbers). Correspondingly, numbers were never thought of as solitary entities but rather as members of some larger communal entity.[11]

Once all the stones in the first row were filled, you would be left with five fingers and you could substitute that for a stone in the second row, representing a hand and being worth 5. This has a linguistic basis. In Quechua "the fingers represent an important set of tools not only for counting but also for naming and organizing ordinal relations."[12] The squares in the second row each have two holes, corresponding with two hands. Once again, the notion of pairing two hands together is a very important one for the Incans and could have even given rise to the Incan use of the decimal system. Two hands together represent 5+5, or 10, the base of the decimal system. The pairing together of two hands in Incan culture is known as the unity of "*khalluntine*." Once each square had two stones, that square would be complete and would represent all ten fingers of one person. Perhaps when all the five squares of the second row were filled, it could represent a complete group of five people (perhaps a family unit?). A full second row would value 50 and could be substituted by a single stone in the third row with value 50.

[11] In the indigenous language Quechua, the numbers one through five related to a family unit. When counting corn for example, the number one referred to a mother and the numbers that followed were all thought of as her offspring. chuqllu (mom), apaña (offspring), iskay apaña (second offspring), kinsa apaña (third offspring), etc. Taken from Urton, *The Social Life of Numbers*, 86.

[12] Ibid., page 73

Arriving at a complete grouping of 50 has significance for the Incans as well. For example, when measuring things, it is always more desirable to arrive at the 50 mark. Consider this Quechua saying for example: "The needle of the balance is still not showing 50 kilos; add a sombrero full of potatoes more."[13]

While this method might not be the most straightforward in terms of calculating values, having a method grounded in cultural tendencies might actually make calculations easier. For example, even though the values of the stones (1, 5, 50, and 750) are row-dependent, perhaps they can be used with ease if they represent a cultural manifestation (example: a finger, a hand, a person, a family, etc.) The previous proposals all make sense logically but are devoid of meaning because they have no social or cultural ties. These interpretations would be readily accepted in a western society that always perceives numbers in an abstract realm, but this is the underlying assumption that contradicts the very way the Incans viewed numbers. For the Incans, "numbers are conceptualized in terms of social- especially family and kinship- roles and relations."[14] Numbers are not simply symbolic representations but rather integral components of culture and societal relations; without these ties there can be no counting device in the first place.

Glynn's Solution

William Burns Glynn definitely took these cultural characteristics into consideration when forming his ideas about this topic. Glynn was a textile engineer from England who had an incredible knack for both mathematics and Quechua. He spent a considerable amount of time working in Peru and was more than familiar with the Incan civilization. His work entitled "*La Tabla de Calculo de los Incas*" published in 1981 presents a very convincing argument about the method of use of the Incan abacus. Glynn starts his analysis with the same quotation from de

[13] Ibid., page 151
[14] Ibid., page 13

Ayala stated at length above. From this quotation various basic, but nonetheless fundamental, conclusions are drawn: (1) the decimal system is used, (2) words are assigned to the different numbers, (3) a calculating tablet is indeed used, and (4) stones or other counters were used to represent the numerical values.

Glynn sets up the abacus so that the long side of the abacus with the five-hole squares are closest to the operator. (See Figure 3). The rightmost column is reserved for the ones digits, followed by the tens, hundreds, thousands, and ten-thousands in order going from right to left, similar to the place notation system used in modern western mathematics. Each column has a total of 11 holes with the top hole being "*la memoria*"- the "memory." The introduction of the "memory" hole is rather revolutionary because it allows for the decimal system to be in place even though each column has 11 holes instead of ten. When counting, the holes from each row are filled (from bottom to top) and once all ten holes are full, they are removed from the board and one stone is placed in the memory hole. The stone can then easily be moved into the second column because it represents the number ten. Having this memory placeholder helps eliminate human error because it temporarily holds the value while the stones are being removed. The operator doesn't need to store that information in his own memory and the operation becomes more mechanic. Once the stones are removed, the memory stone can simply be moved into the next column to the left. This set-up allows for addition, subtraction, multiplication, and division.

Another breakthrough is that Glynn's analysis is the only one to find a purpose behind the hole structure. Each column is composed of squares that break down into one, two, three, and five holes. These numbers represent the infamous Fibonacci Sequence. Utilizing this sequence greatly reduces the number of steps needed to perform a multiplication or division. For example, if you were to multiply 150 x 70, instead of adding 150 to itself 70 times, these

Fibonacci numbers could be used to simplify the problem into only two additions. This trick is used in the interpretation of *la yupana* that follow; a more detailed explanation can be found in the section on multiplication.

Improved Abacus Methods

Figure 3, from Cahill.

While the Burns proposal is the most probable, there are a few discrepancies and further commentary is needed. The first main incongruity deals with the way the abacus itself is situated. Burns argues that the abacus should not be situated as it is in the picture, but rather with the long side (of squares of five holes) towards the *quipucamayoc.* He uses the *Margarita Philosophica* by Gregorius Reisch as the basis for this claim. This 16th century encyclopedia has European origins and was used particularly in Germany. In this work, there is a picture of a counting device, situated so that the long side is closest to the operator. (See Figure 9). Burns writes that "the rules of Industrial Engineering in so far as they refer to the proper way to situate a work, suggest that the longest side should be placed by the operator. Likewise, the squares with more circles should be closer to the operator to avoid unnecessarily large movements."[15] While this might be the case in many European industrial settings, it is not likely that the Incans positioned the abacus in that way. This line of thought is clearly influenced by the western emphasis on efficiency as well as the way numbers are written in the western world: with the ones, tens, hundreds, etc. moving from right to left. While this correspondence with western practices doesn't necessarily disprove the theory, it disregards the Incan practices that lend themselves to a more probable positioning of the abacus. The Incas kept track of numbers on the *quipus* with the ones, tens, hundreds, etc. starting from the bottom of the cord and working

[15] Burns, page 60, my translation

upwards. It only makes sense to model the abacus the same exact way. This allows the numbers to be taken directly (and very easily) from (and transcribed onto) the *quipu* cords. There would be a direct correlation between the abacus and the *quipus.* This also coincides with what Acosta writes about the Incan mode of writing: "Their mode of writing wasn't in a straight line, but rather from top to bottom."[16] It seems only logical that the abacus be used in the very manner it appears in the drawing of De Ayala- and more exact evidence would be required in order to rotate the image as Burns so suggests. The remaining part of this essay will use the position of the abacus and row labels as shown in Figure 5 below.

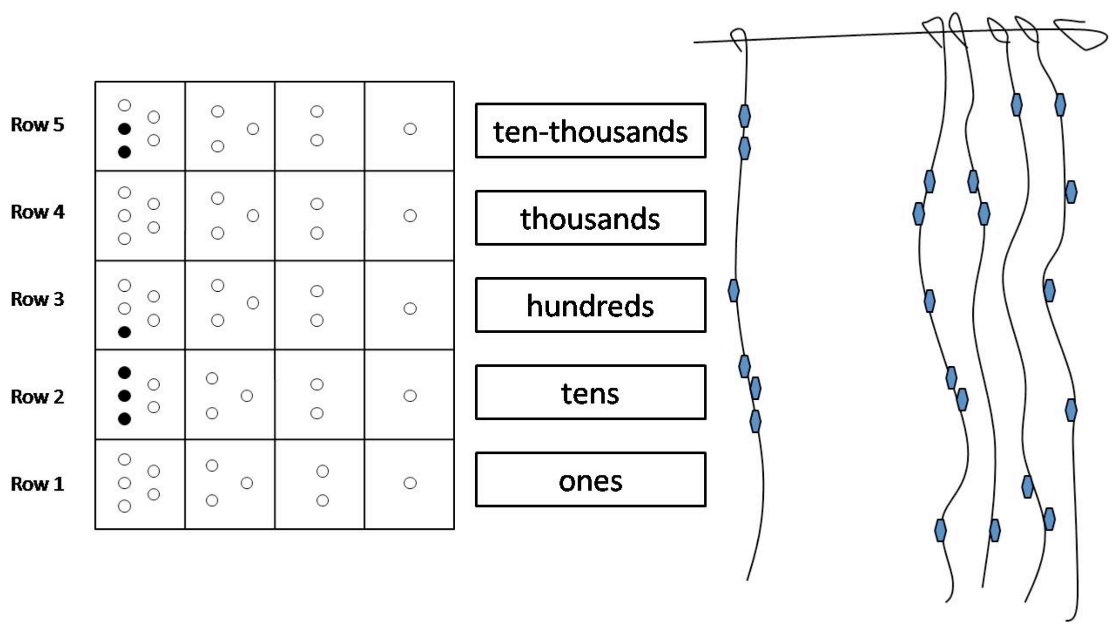


Figure 4 The abacus when vertical, corresponds directly with how quipus are read and used. The abacus and the first cord of the quipu are displaying the number 20,130.

[16] Acosta, page 292, author translation

There is also a better way to consider the decomposition technique used in the multiplication and division problems. Burns proposes that the decomposition can easily be seen as a matter of looking at the stones and separating them into groupings of one, two, three, and five. Instead of resorting to a visualization of the stones, the abacus itself can be used to more exactly and effectively determine the decomposition of any number into its corresponding factors. (The purpose of this will be made clear in the following examples.) The stones could be placed on the left side and translated to the right until the squares completely filled. (This process is also very consistent with the cultural aim of "completion" discussed earlier). For example, when displaying the number 47 on the abacus (see Figure 6), stones would be moved to the right, with three stones remaining in the in the second square and the last stone completing the last square. The seven stones in the ones row already complete the first square with five holes and the remaining stone would fill the square of two holes. Once the number is displayed on the board this way, it is very obvious to see that the number can be broken down into 30, 10 (the three and one stones in the second row, because in the tens row, take on the values 1 x 10 = 30 and 1 x 10 = 10 respectively), 5, and 2. The abacus structure lends itself to immediately finding the decomposition of any number; this is more effective than any visual technique.

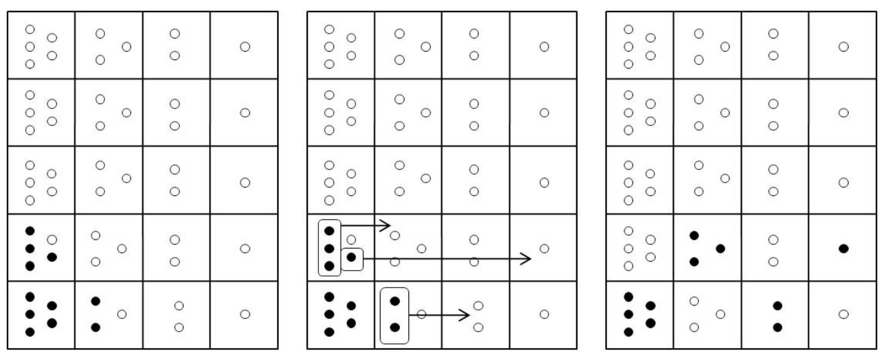


Figure 5 Showing that the abacus can be used to easily demonstrate that 47 = 30 + 10 + 5 + 2

One last addition that must be made deals with the effectiveness of the "memory hole." The stone placed in the memory hole could remain there until all the remaining stones were added to the abacus. In this way, the person performing the calculation could place all stones on the abacus in one fluid motion, eliminating possible error. The final step would be to move the stone from the memory hole into a hole from the row above. Saving this step for the end would indeed serve the purpose of a "memory holder," freeing up the operator to continue the calculation and the stone addition without any interruptions of motion or train of thought.

Learning from previous methods and taking into consideration Glynn's work, we are able to devise an interpretation of the abacus that proves most logical considering the information available. The following operations will be executed using the author's original interpretation of the counting device. They follow the basic structure of the operations outlined in *La Tabla de Calculo* de los Incas, but take into consideration the improvements suggested above and are simplified considerably (especially in the division problem) to the benefit of the operator of the abacus.

Addition

Addition (as executed in Figure 7) is the most straightforward of all calculations, and probably one of the most common operations. One of the summands is placed on *la yupana* while the other summand is placed outside *la yupana*, to the right of the corresponding rows (A). The stones outside the abacus in the ones row are added to *la yupana* first (B), using the memory stone as needed. Whenever the total number of stones in any given row reaches ten (hence the 5-square, 3-square, and 2-square are all full) then those stones can be removed and one stone placed in the single hole (the memory hole) on the right (C). The rest of the stones can then be added to the now-empty holes of that same row (D). After all the stones have been added, the stone from the memory hole can be shifted up to occupy a space in the next row up (E) and the process continues by adding the summand from that next row. All the place values are computed in the same fashion, always moving from the bottom to top. Once all the summand stones on the right side of the abacus have been worked into the calculation, the number remaining on *la yupana* is the answer, in our case 516.

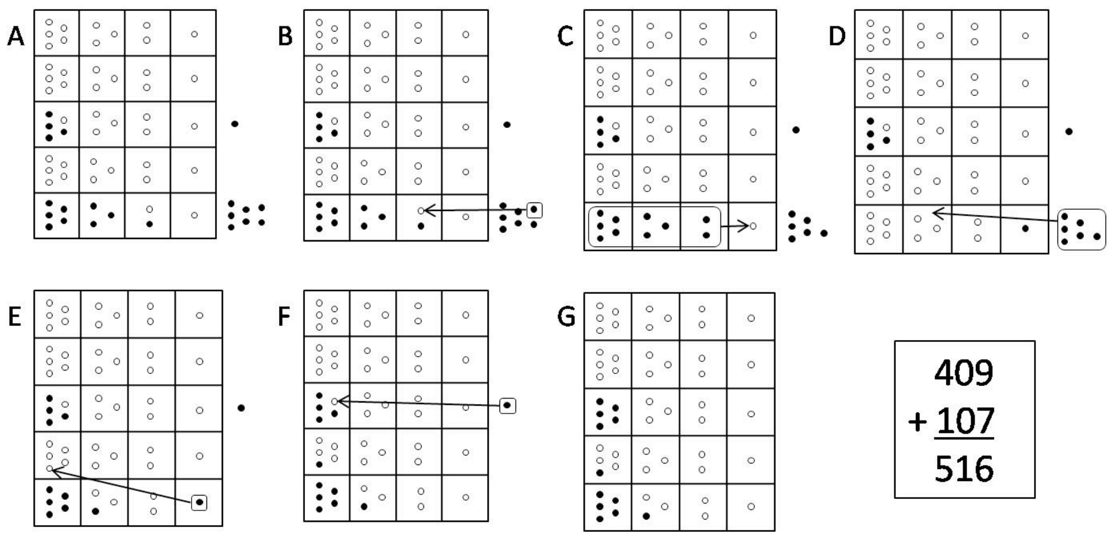


Figure 6

Sub traction

Subtraction is a similar computation, but the steps are done in reverse. (Refer to Figure 8 to follow along with the written explanation of 715-260). Subtraction would have played just as an important role as addition in order to rectify imbalances and keep track of the accounting transactions. In this case, the larger number is placed on the abacus and the number being subtracted is placed outside the abacus (A), to the right of the appropriate rows. (Refer to Figure 7). This time the calculation is performed from top to bottom. The stones are removed from the abacus and placed to the left of the tablet (B-C). Stones are removed until the number of stones on both sides of the abacus match. If there are not enough stones, then a stone from a superior row (the row directly above) is moved to the memory space of the row just below it (D). That memory stone can be converted into ten stones which occupy the holes in that row (E) and the stone removal can continue (F). Stones are transferred and removed until the number of stones displayed to the right of the board and the number of stones removed (represented by the stones that are found to the left the board) are the same (G). At this point, you have a visual confirmation that the calculation is completed; the number that remains on the abacus is the result of the subtraction, this time being 455, see Figure 8.

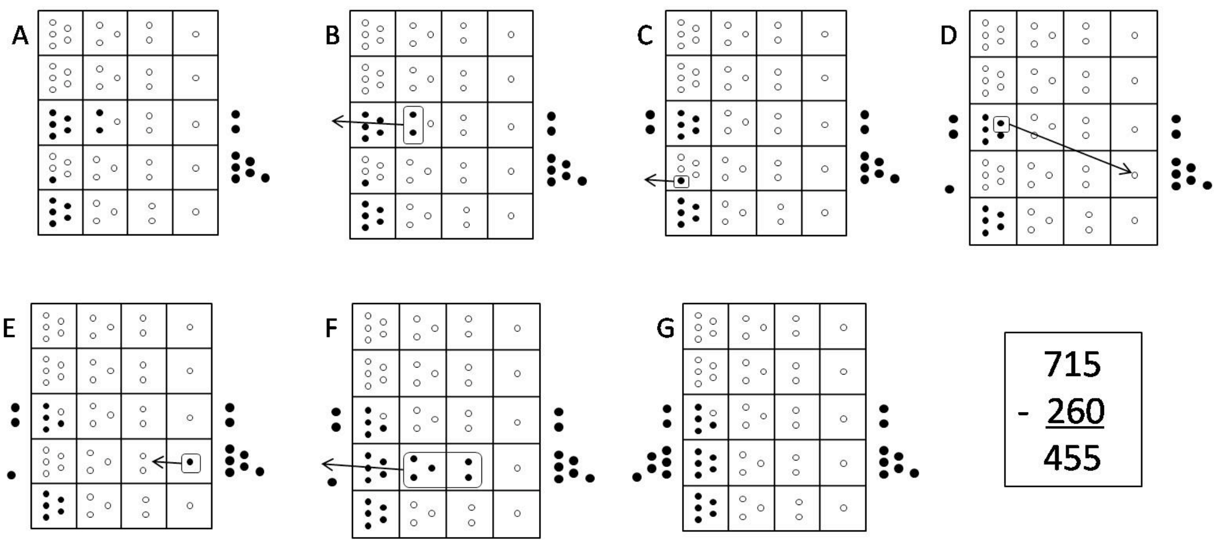
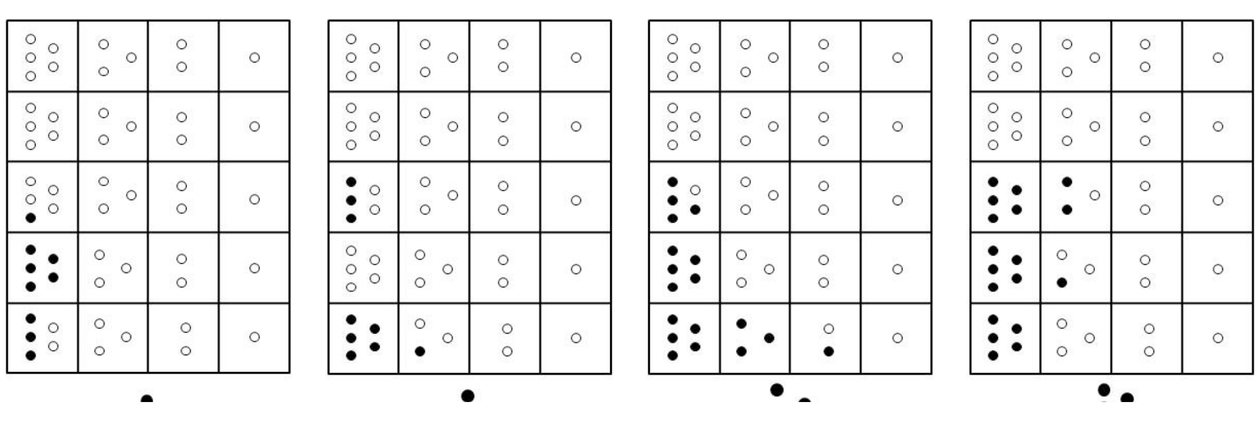


Figure 7

Multiplication

Using the abacus to multiply is a bit more involved, but the genius of the operation lies in the fact that it makes use of the hole structure of the abacus (i.e. the squares having 1, 2, 3, and 5 holes). We notice that the presence of the Fibonacci Sequence is not only an interesting observation, but that it has practical significance in this interpretation of the abacus. The Fibonacci sequence is constructed by summing the two previous numbers in the sequence. This



aids in multiplication because it is no longer necessary to add the same number together as many

Figure 9, the preparation tables for the multiplication operation, representing 153x1=153, 153x2=306, 153x3=459, 153x5=765.

times as the value of the multiplier; we can take advantage of the progression developed by adding together partial products.

Multiplication requires many preliminary steps before the calculation itself is carried out. Perhaps this explains the fact that there were always as least a minimum of two *quipucamayocs*. Each *quipucamayoc* could help perform the preliminary steps on his own tablet. Let's look at the example 153 x 47. In order to prepare, the following information would be collected about the greater of the multipliers: 153 x 1 = 153, 153 x 2 = 306, 153 x 3 = 459, and 153 x 5 = 765. These calculations could be very easily resolved. 153 times 1 is obvious: 153. 153 times 2 is just that number added together twice. 153 times 3 is the two previous results added together and so forth. Each of the results could easily be displayed on a separate tablet (like those shown in Figure 9).

The next important observation is that the second multiplier can always be broken down into a sum of numbers that are easier to work with. Every multiplier can be decomposed into a sum of Fibonacci numbers, including various powers of ten. For example, the number 47 could be thought of as (30+10+5+2). To review, after the counters are placed on the tablet, they can very easily be moved around in order to determine the factors that can be summed together to arrive at the desired result. In accordance with the Incan principle of completion, any counters that are in a square but not able to complete it, are translated to the square to the right, until they are able to completely fill the square. Once all counters are located in complete squares, the factors of the decomposition are simply the value of each square. (Refer back to Figure 6 for clarification).

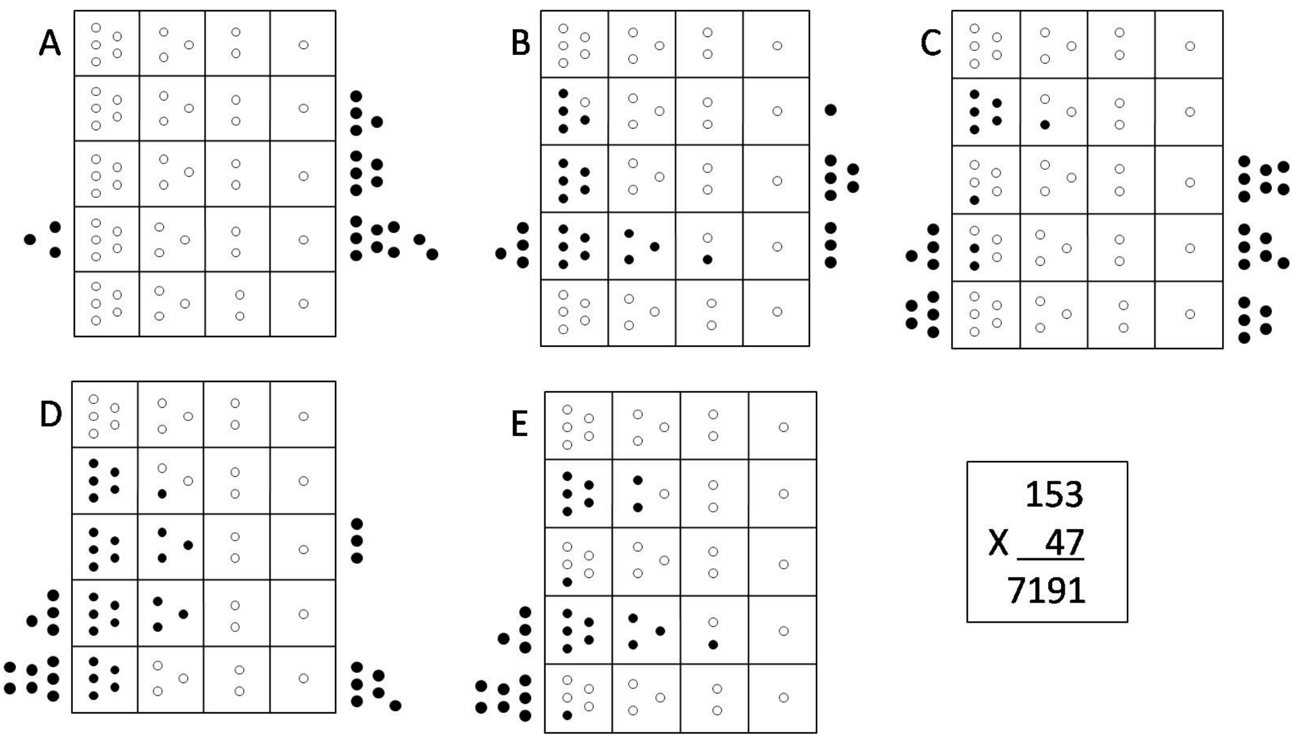


Figure 10

After the second multiplier is broken down into ones, twos, threes, and fives, the problem can be solved through repeated addition. Instead of adding 153 forty-seven times, we can simply add together four number (the number of additions matches the number of terms in the decomposition); the problem becomes greatly simplified and efficient. (See Figure 10). Recalling the decomposition already established previously (47 = 30 + 10 + 5 + 2) we start with the largest factor: 30. We find the number that corresponds with the partial product of the third and multiply it by a power of ten. (153 x 3 = 459 x 10 = 4590). This number is placed to the right of the abacus, and three stones are placed to the left of the second row (A) to represent the factor that has already been taken into consideration. The next step is to take the number that corresponds with the factor 10. (153 x 1 = 153 x 10 = 1530). 1530 stones are placed to the right of the abacus while one stone is placed in the tens row to the left of the abacus (B). The summation is performed. Next, the number corresponding to the factor five is added into the equation. (153 x 5 =) 765 is added to the 4590 + 1530 already found on the *quipus*. Five stones are then placed to the left of the first row (C). Finally 278 is added to the abacus and two stones are placed to the left of the last row to denote the addition of the last factor (D). At this point, you are left with the number 47 on the left side of la yupana confirming the fact that the entire number was factored into the calculation. The number that remains on the abacus is the product of the multiplication (E). In this case, that number is 7191 (4590 + 1530 + 765 + 306 = 7191).

Division

Division is perhaps the most difficult operation, but is more easily understood with the aid of a counting device like the Incan abacus. Just like with multiplication, the first part of the process is to make a table of multiples of the divisor. Let's use the same tables that were prepared for the multiplication problem (Figure 6) and look at the problem 7191/153. The dividend 7191 is placed on the abacus. (Refer to Figure 11). Working top to bottom (like in the case with subtraction), we look at the value in the highest row- in this case there is a 7 in the thousands row. We must refer back to the preparation tables in order to find something that is equal to or less than that value. None of the values are less than 7, in fact the smallest value shown is 153 x 1 = 153. We see right away that we must move down a row but using the hundreds row as our starting place we still only have the value 71 which is not sufficient. Moving down again we stop at the tens row and see that if the "decimal point" is placed on this row then we are looking at the number 709. We can now go back again to the preparation tables to find the highest value represented that does not surpass 709. The last value (765) is too large and we settle for the third tern- 459. Three stones are placed on the right side of the abacus (in-line with the tens row) and 459 stones are removed from the abacus and placed on the left side (A). The removal of stones follows the same rules as a traditional subtraction problem. In some cases, a stone from a higher row will have to be converted into ten stones of a lower row (B). After the stones on the right side and the stones on the left side perfectly match the respective term of the preparation table (C matches the third *yupana* of the preparation *yupanas* of Figure 6), the subtracted stones on the left side of *la yupana* can be removed in order to simplify the computation. The number left on the abacus is 260 (we are still using the tens position as our point of reference). Are any of the values from Figure 8 260 or less? The first factor of 153 is and so it can be subtracted as well. Again, one stone (representing that we are using the first factor) is placed to the right of the abacus while 153 is subtracted from the abacus and placed to the left (D-E). The number that remains is 107 and the subtracted stones on the left side of *la yupana* can be cleared away (F). 107 is smaller than all the factors of 153 so we must move our reference point to the ones position. From that vantage point we are left with 1071 and can subtract the largest factor, 153 x 5 = 765. Five stones are placed on the right side of the abacus (in-line with the first row) and 765 must be subtracted (G). After subtracting the single stone in the first row, that row is vacated and so the stone in the thousands-row must move down a row and be converted into 10 one-hundreds. We are now able to subtract the remaining stones, by moving them to the left of the abacus and adding a five to the right of the ones row (H). Once the stones are subtracted and match the number of stones as represented in the preliminary table of values (of the last table, correlated with the 5-factor) they can be removed from the work space (I). It is now readily seen that the 306 stones left on the abacus correspond with the second factor of 153, 153 x 2 = 306. Therefore the last factor of two is placed on the right side of the abacus in-line with the ones row and the 306 stones are removed to the left side (J).

Once the abacus is empty, the quotient remains on the right side of *la yupana* (K). In our example, the answer is 47 because we had placed three stones in the tens row, one stone in the tens row, five in the ones row and finally two more in the ones row. (30 + 10 + 5 + 2 = 47). This answer is expected from the results of the previous multiplication problem but this example shows how each step of the operation is computed. Additional abacus movements are needed in between some of the steps shown in Figure 11 but the main steps are all illustrated (some of the steps that involve changing a stone from a higher row into ten stones in the lower row via the memory hole are taken as understood because they are shown in previous examples). While this operation does seem rather complicated, it is simply a repeated subtraction problem just as multiplication is a repeated addition problem. Once the addition and subtraction methods are understood and the basic structure of the multiplication and division problems, any computation can be executed rather simply and straightforwardly.

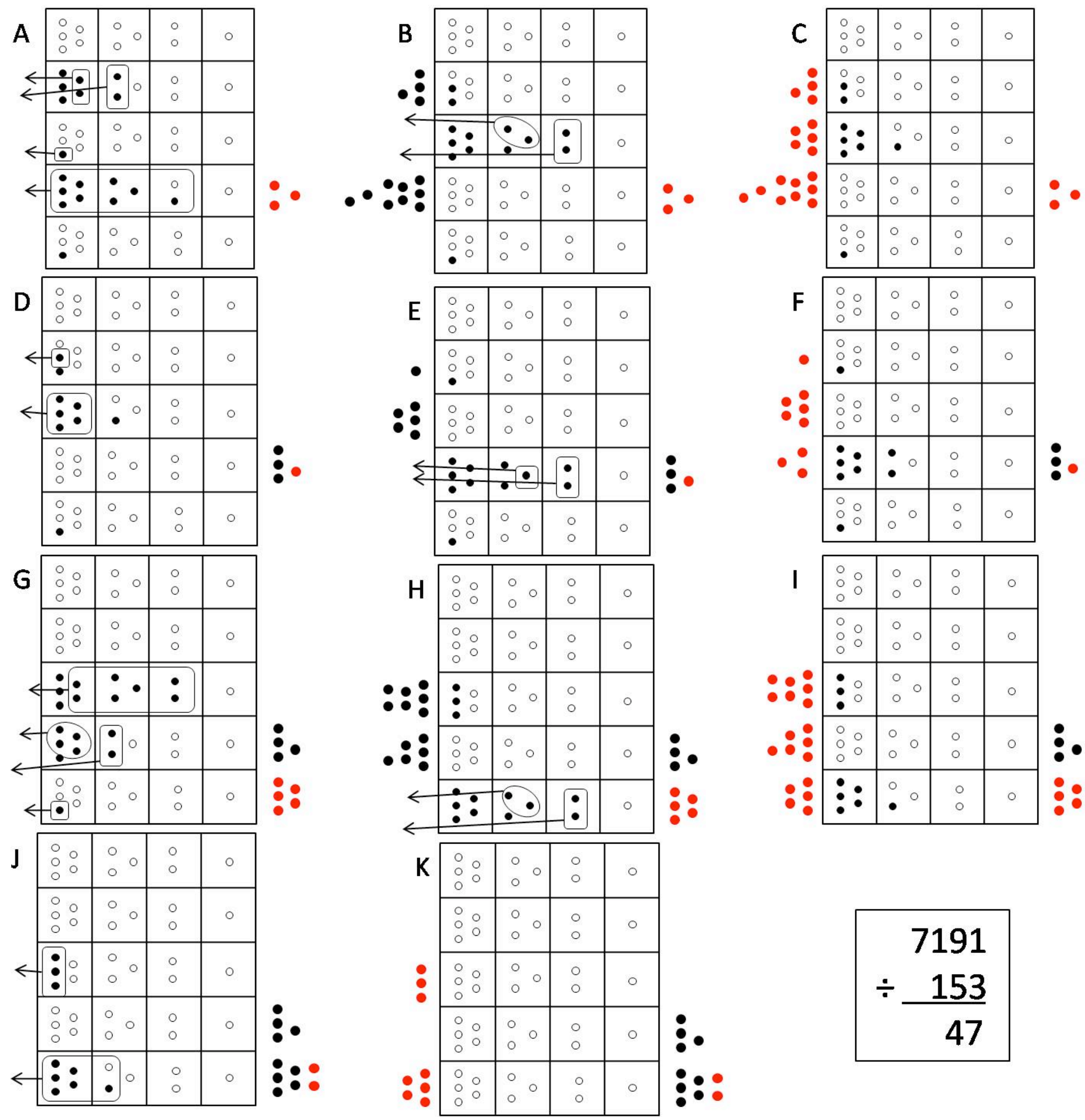


Figure11

Linguistic Link

Because *la yupana* is a cultural artifact as well as a counting device, it should have a cultural explanation as well as a mathematical one. The Burns interpretation is more than sufficient to show that everyday calculations can effectively be carried out, but there must be cultural evidence that supports the same interpretation as well. The Andean-linguistic expert Guido Alfredo Pilares Casas provides the missing link. In his article entitled "*Los Sistemas Numericos Del Quechua y el Aymara,*" Casas discovers a connection between *la yupana* and the structure of the ancient Andean language of Aymara. This language was endemic to the very region in which the abacus was found, and Gerald Taylor suggests that Guaman Poma de Ayala the chronicler and creator of *la yupana* image itself spoke some form of Aymara that is now extinct. Casas considers from a linguistic perspective the distribution of holes into groups of five, three, two, and one. The answer relates nicely to the quinario-decimal structure of the Aymara language. While in the basic decimal system, each number builds off a base of ten, twenty, thirty, etc., in the quinario-decimal system each number builds off multiples of five as well as multiples of ten.

In this way the word five is denoted in Aymara by "*qallqu*." The three stones in the next square can all be derived from this value of five. Six is known as "*maqallqu*" ("*ma*" + "*qallqu*" = 1 + 5), seven as "*paqallqu*" ("*pa*" + "*qallqu*" = 2 + 5), and eight as "*kimsaqallqu*" (*kimsa*" + "*qallqu*" = 3 + 5). The next two stones are grouped together and can be derived from the value of ten. Nine is known as "*llatunka*" in Aymara, meaning "ten minus one." Ten is simply "*tunka*." Once these stones are all combined and placed in the memory hole, there is another word in the Aymara language that means ten. "*Matunka*" which literally means "one ten." The numbers in the Aymara language coincide with the structure of the Incan abacus perfectly. This would make the instrument all the more accessible to the Incan people; this very same interpretation of *la yupana* is still used today and is part of the curriculum in the rural, bilingual schools of Peru.

Conclusion

While the above interpretation of the Incan abacus is the most accurate and comprehensive to-date, the lack of primary resources on the topic renders it impossible to know exactly how each minor movement was made. While the overall computation can be determined based on the mathematical and cultural clues, some of the smaller steps as presented could actually be performed differently. For example, in the division operation, the stones wouldn't necessarily have to be cleared away after each repeated subtraction. Also, the addition and subtraction could be computed in another order (it wouldn't matter whether the movement was from top to bottom or bottom to top). Likewise, since multiplication is a repeated addition problem, it makes no difference the order in which the additions are executed. Other possibilities include switching the side of the abacus that the stones are placed or even using additional *yupanas* on either side of the main *yupana* in order to store the stones. While these distinctions appear arbitrary, they are not completely so because the operations chosen in this analysis were those which were of greatest ease-of-use and clarity. While distinctions can be made, it is important that unity is remained throughout.

La yupana was a very important instrument in the Incan empire and not nearly enough information has been gathered about it. From the few primary and secondary sources that exist, we can reach some basic conclusions about how this instrument was used. Burns made significant progress and was the first to give an adequate explanation of the abacus, but was missing some key pieces of the puzzle. After comparing his conclusions with additional cultural information, we were able to enhance our understanding. *La yupana* was most likely used in the same way as it is seen in the Ayala's drawing; this renders a direct correlation between *la yupana* and *el quipus*. The tablet itself and its structure can be very easily used to factor any number.

Another very pertinent observation is that the structure of the counting board directly correlates with the structure of and is reflected in the Aymara language itself.

It is also very fascinating to point out that such an interpretation is similar to that of the Chinese abacus. The Chinese abacus has beads of values 1 and 5. The numbers 6, 7, 8, and 9 are established by adding “ones” to the “five” bead. This abacus operates in a quinario-decimal system just like *la yupana*. These two counting instruments were developed independently of one another and the similarities found between the two only make the argument and interpretation of *la yupana* proposed in this essay all the more credible. Even though we are far from exhausting this discussion and decoding the complete mystery behind the Incan abacus, we are putting the stones in place for building a solid foundation and a better understanding of the mathematical practices of the Incas.

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